

Ground states in relatively bounded quantum perturbations of classical lattice systems

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Abstract. We consider ground states in relatively bounded quantum perturbations of classical lattice models. We prove general results about such perturbations (existence of the spectral gap, exponential decay of truncated correlations, analyticity of the ground state), and also prove that in particular the AKLT model belongs to this class if viewed at large enough scale. This immediately implies a general perturbation theory about this model.

Key words: ground state, relative boundedness, AKLT model, cluster expansion.

1 Introduction and results

It is generally expected that if a ground state of a quantum lattice system is in a non-critical regime characterized by the presence of a spectral gap and exponential decay of truncated correlations, then the system remains in this phase under sufficiently weak perturbations of a general form. Relevant rigorous results are now available in the case of weak quantum perturbations of some classical models [4, 8, 16, 17, 18, 22, 23, 29, 30]. Most of these results concern perturbations which are bounded and small in the norm sense. However, Kennedy and Tasaki obtained in [17] general results for perturbations, which are only relatively bounded, in some special sense, w.r.t. the classical Hamiltonian. Moreover, using a special transformation of the Hamiltonian, they applied this perturbation theory to the dimerized AKLT model, which is a genuinely quantum $SU(2)$ -invariant model. The type of relative boundedness they used does not, however, seem to allow an extension of their result to the non-dimerized, fully translation invariant case. In this paper we consider perturbations relatively bounded in the quadratic form sense, which appears

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to fit naturally in this and some other contexts. We prove general results for gapped classical models with a simple ground state and then apply them to the non-dimerized AKLT model.

We consider a quantum “spin” system on the lattice \mathbb{Z}^ν . Throughout the paper we consider only translation invariant interactions. Each site $x \in \mathbb{Z}^\nu$ is equipped with a Hilbert space \mathcal{H}_x , possibly infinite-dimensional. In the sequel we use the notation

$$\mathcal{H}_\Lambda \equiv \bigotimes_{x \in \Lambda} \mathcal{H}_x$$

for Hilbert spaces, corresponding to finite subsets of the lattice. We assume that \mathcal{H}_x has a preferred vector denoted Ω_x . The corresponding product state will be denoted by $\Omega_{\Lambda,0}$:

$$\Omega_{\Lambda,0} \equiv \bigotimes_{x \in \Lambda} \Omega_x.$$

Also, we fix some finite set $\Lambda_0 \subset \mathbb{Z}^\nu$, which will be the interaction range. The (formal) Hamiltonian has the form

$$H = H_0 + \Phi,$$

where H_0 is the classical part and Φ the perturbation. The classical Hamiltonian H_0 is given as

$$H_0 = \sum_{x \in \mathbb{Z}^\nu} h_x.$$

Here h_x is a self-adjoint, possibly unbounded operator acting on $\mathcal{H}_{\Lambda_0+x}$, where Λ_0+x is a shift of Λ_0 . The Hamiltonian H_0 is classical in the following sense. If \mathcal{H}_x is finite dimensional, then we assume that in each \mathcal{H}_x there is an orthogonal basis containing Ω_x and such that the product basis in $\mathcal{H}_{\Lambda_0+x}$ diagonalizes h_x . We extend in a natural way this assumption to the case of infinite dimensional \mathcal{H}_x by assuming that for each \mathcal{H}_x an orthogonal partition of unity, containing the projection onto Ω_x , is given, and h_x is a function of the product partition in $\mathcal{H}_{\Lambda_0+x}$. Furthermore, we assume that Ω_{Λ_0+x} is a non-degenerate gapped ground state of h_x :

$$h_x \Omega_{\Lambda_0+x,0} = 0, \quad h_x|_{\mathcal{H}_{\Lambda_0+x} \ominus \Omega_{\Lambda_0+x,0}} \geq \mathbf{1}. \quad (1)$$

Now we describe the perturbation. It is given by

$$\Phi = \sum_{x \in \mathbb{Z}^\nu} \phi_x,$$

where ϕ_x is a (possibly unbounded) symmetric quadratic form on $\mathcal{H}_{\Lambda_0+x}$, bounded relative to the quadratic form corresponding to h_x , i.e. the domain of ϕ_x contains $\text{Dom}(h_x^{1/2})$ and

$$|\phi_x(v, v)| \leq \alpha \|h_x^{1/2} v\|^2 + \beta \|v\|^2, \quad v \in \text{Dom}(h_x^{1/2}) \quad (2)$$

with some α, β . We assume that $\alpha < 1$. The form ϕ_x actually need not be closed and generated by an operator, though in all examples we consider it is.

If Λ is a finite volume and

$$H_{\Lambda,0} = \sum_{x:\Lambda_0+x\subset\Lambda} h_x, \quad \Phi_\Lambda = \sum_{x:\Lambda_0+x\subset\Lambda} \phi_x,$$

then Φ_Λ is again bounded relative to $H_{\Lambda,0}$ with the same α , because, clearly, $\text{Dom}(H_{\Lambda,0}^{1/2}) \subset \text{Dom}(\Phi_\Lambda)$ and, by adding up (2),

$$|\Phi_\Lambda(v, v)| \leq \alpha \|H_{\Lambda,0}^{1/2} v\|^2 + |\Lambda| \beta \|v\|^2, \quad v \in \text{Dom}(H_{\Lambda,0}^{1/2}).$$

It follows from the KLMN theorem that $H_\Lambda = H_{\Lambda,0} + \Phi_\Lambda$ is a well-defined self-adjoint operator, defined by its quadratic form [14, 25]. Throughout the paper unbounded operators will appear only as relatively bounded perturbations of positive operators in the quadratic form sense, so, in order not to complicate arguments and keep the notation simple, we will typically not distinguish between operators and corresponding quadratic forms.

We will assume now for simplicity that Λ is a cubic volume with periodic boundary conditions. Clearly, in this case $\Omega_{\Lambda,0}$ is a non-degenerate ground state of $H_{\Lambda,0}$ with a spectral gap:

$$H_{\Lambda,0}\Omega_{\Lambda,0} = 0, \quad H_{\Lambda,0}|_{\mathcal{H}_\Lambda \ominus \Omega_{\Lambda,0}} \geq |\Lambda_0| \mathbf{1}.$$

The following result is a perturbation theory for the ground state in the case of small α and β .

Theorem 1. *There exist positive α and β , depending only on the dimension ν and the interaction range Λ_0 , such that if condition (2) holds with these α, β , then:*

1) H_Λ has a non-degenerate gapped ground state Ω_Λ :

$$H_\Lambda \Omega_\Lambda = E_\Lambda \Omega_\Lambda,$$

and for some independent of Λ positive γ

$$H_\Lambda|_{\mathcal{H}_\Lambda \ominus \Omega_\Lambda} \geq (E_\Lambda + \gamma) \mathbf{1}.$$

2) There exists a thermodynamic weak*-limit of the ground states Ω_Λ :

$$\langle A \Omega_\Lambda, \Omega_\Lambda \rangle \xrightarrow{\Lambda \nearrow \mathbb{Z}^\nu} \omega(A), \quad A \in \cup_{|\Lambda| < \infty} \mathcal{B}(\mathcal{H}_\Lambda),$$

where $\mathcal{B}(\mathcal{H}_\Lambda)$ is the algebra of bounded operators in \mathcal{H}_Λ .

3) There is an exponential decay of correlations in the infinite volume ground state ω : for some positive c and $\epsilon < 1$

$$|\omega(A_1 A_2) - \omega(A_1)\omega(A_2)| \leq c^{|A_1|+|A_2|} \epsilon^{\text{dist}(A_1, A_2)} \|A_1\| \|A_2\|, \quad A_i \in \mathcal{B}(\mathcal{H}_{\Lambda_i}).$$

4) If within the allowed range of perturbations the terms ϕ_x (or the resolvents $(h_x + \phi_x - z)^{-1}$ in the case of unbounded perturbations) depend analytically on some parameters, then the ground state ω is also weakly* analytic in these parameters (i.e. for any local observable A its expectation $\omega(A)$ is analytic).

Example 1 (anharmonic quantum crystal model). Let $\mathcal{H}_x = L_2(\mathbb{R}^d, dq)$ and

$$H = \sum_x (-\Delta_x + V_1(q_x)) + \lambda \sum_{|x-y|=1} V_2(q_x, q_y).$$

Suppose that $V_1(q) \rightarrow +\infty$ as $q \rightarrow \infty$. In this case $-\Delta + V_1$ has a discrete spectrum with a non-degenerate ground state. Since $-\Delta \geq 0$, we see that if for some c_1, c_2

$$|V_2(q_x, q_y)| \leq c_1(V_1(q_x) + V_1(q_y)) + c_2, \quad \forall q_x, q_y,$$

then for sufficiently small coupling constant λ the operator H_Λ is well-defined by the KLMN theorem, and Theorem 1 applies.

The next theorem extends the perturbation theory to all $\alpha \in (0, 1)$ at the cost of a slightly more stringent assumption about the perturbation. We replace (2) with the following stronger assumption:

$$\phi_x = \phi_x^{(r)} + \phi_x^{(b)}, \tag{3}$$

where $\phi_x^{(r)}$ is the “purely relatively bounded” part of the perturbation:

$$|\phi_x^{(r)}(v, v)| \leq \alpha \|h_x^{1/2} v\|^2, \tag{4}$$

and $\phi_x^{(b)}$ is the bounded part:

$$\|\phi_x^{(b)}\| \leq \beta. \tag{5}$$

In particular, (4) and (1) imply that $\phi_x^{(r)} \Omega_{\Lambda_0+x,0} = 0$ if $\phi_x^{(r)}$ is viewed as an operator (more precisely, $\phi_x^{(r)}(v, \Omega_{\Lambda_0+x,0}) = 0$ for all $v \in \text{Dom}(h_x^{1/2})$).

Theorem 2. For any $\varkappa > 1$ there exists $\delta = \delta(\varkappa, \nu, \Lambda_0) > 0$ such that: for any $\alpha \in (0, 1)$, if conditions (3)-(5) are satisfied with this α and $\beta = \delta(1 - \alpha)^{\varkappa(\nu+1)}$, then all conclusions of Theorem 1 hold.

Remark. The assumption (4) can be somewhat relaxed. In fact, what is actually used in the proof of Theorem 2 is not (4) but the weaker condition: for any $I \subset \Lambda$

$$\left| \sum_{x \in I} \phi_x^{(r)}(v, v) \right| \leq \alpha \|H_{\Lambda,0}^{1/2} v\|^2. \quad (6)$$

This is the condition which we will use when we consider the AKLT model.

Example 2. Consider a Hamiltonian

$$H = \sum_x A_x,$$

where A_x is a self-adjoint operator on $\mathcal{H}_{\Lambda_0+x}$ such that

$$A_x \Omega_{\Lambda_0+x,0} = 0, \quad A_x|_{\mathcal{H}_{\Lambda_0+x} \ominus \Omega_{\Lambda_0+x,0}} \geq \mathbf{1}.$$

Clearly, $\otimes_x \Omega_x$ is a ground state of H with a gap $\geq |\Lambda_0|$. We expect that a perturbation theory in the sense of Theorems 1,2 holds at least for general weak bounded perturbations of H . Theorem 2 shows that this is indeed so at least if $\|A\| < \infty$ (A is the operator whose translates A_x 's are). Indeed, consider a finite range perturbation $\sum_x \psi_x$ with small $\|\psi_x\|$. By some rearrangement of terms in H , we may assume without loss of generality that ψ_x acts on $\mathcal{H}_{\Lambda_0+x}$. Now, let $A_x = h_x + \phi_x^{(r)}$, where

$$h_x = \|A\| P_{\mathcal{H}_{\Lambda_0+x} \ominus \Omega_{\Lambda_0+x,0}}, \quad \phi_x^{(r)} = A_x - \|A\| P_{\mathcal{H}_{\Lambda_0+x} \ominus \Omega_{\Lambda_0+x,0}}.$$

Here and in the sequel P_X stands for the projector onto X . It follows that $\sum_x h_x$ is a classical Hamiltonian satisfying our assumptions and, by the spectral gap condition on A , $\sum_x \phi_x^{(r)}$ is its relatively bounded perturbation so that (4) holds with $\alpha = (\|A\| - 1)/\|A\| < 1$. We consider now ψ_x as $\phi_x^{(b)}$, and then Theorem 2 applies.

Now we describe the application of Theorem 2 to the AKLT model. This model was introduced by Affleck *et al.* [2, 3] as the first rigorous example of a system in the Haldane phase ([12, 13], see [1] for a review of the Haldane conjecture). It is a spin-1 chain with the translation-invariant nearest-neighbor isotropic interaction

$$H = \sum_{k \in \mathbb{Z}} P^{(2)}(\mathbf{S}_k + \mathbf{S}_{k+1}) \equiv \sum_{k \in \mathbb{Z}} (\mathbf{S}_k \cdot \mathbf{S}_{k+1}/2 + (\mathbf{S}_k \cdot \mathbf{S}_{k+1})^2/6 + 1/3),$$

where \mathbf{S}_k is a spin-1 vector at site k , and $P^{(2)}(\mathbf{S}_k + \mathbf{S}_{k+1})$ is the projector onto the subspace where $\mathbf{S}_k + \mathbf{S}_{k+1}$ has total spin 2. The AKLT model has

a unique gapped ground state ω minimizing the energy of each term in the interaction:

$$\omega(P^{(2)}(\mathbf{S}_k + \mathbf{S}_{k+1})) = 0$$

(a frustration-free ground state). The state ω can be described as a valence-bond-solid state [3] or a finitely correlated state [10, 11]. On a finite chain Λ with periodic boundary conditions the AKLT Hamiltonian H_Λ has a unique frustration-free ground state.

Let $\Phi = \sum_k \phi_k$ be any translation-invariant finite range interaction on the spin-1 chain. We consider the perturbed AKLT model $H + \Phi$, starting, as before, with periodic finite chains Λ . We prove

Theorem 3. *If $\|\phi_k\| \leq \beta$, with some β depending on the range of Φ , then all conclusions of Theorem 1 hold for the perturbed AKLT model $H + \Phi$.*

The main point of the proof is that at large scale the AKLT model is a relatively bounded perturbation of a classical model. This enables us to use Theorem 2. Though in this paper we restrict our attention to the AKLT model only, this property is definitely more general; one can expect some form of it to be generic to non-critical gapped spin systems.

2 Proof of Theorem 1

We follow the standard approach and approximate the ground state with low-temperature states. After time discretization we obtain a cluster expansion, which identifies the model with a low density hard-core gas of excited regions on the space-time lattice [17]. After that all conclusions of Theorem 1 follow in a usual way. Our exposition is, however, rather different technically: we derive necessary cluster estimates using the Schwarz lemma and resolvent expansions instead of the Feynman-Kac formula.

We begin by proving that H_Λ has a gapped ground state. Fix some $t_0 > 0$ and consider the expectation

$$Z_{N,\Lambda} \equiv \langle (e^{-t_0 H_\Lambda})^N \Omega_{\Lambda,0}, \Omega_{\Lambda,0} \rangle, \quad (7)$$

at large $N \in \mathbb{N}$. If Ω_Λ is a non-degenerate ground state of H_Λ with the energy E_Λ and a spectral gap $\geq \gamma$, then

$$Z_{N,\Lambda} = |\langle \Omega_\Lambda, \Omega_{\Lambda,0} \rangle|^2 e^{-t_0 E_\Lambda N} + O(e^{-t_0(E_\Lambda + \gamma)N})$$

and hence, if $\langle \Omega_\Lambda, \Omega_{\Lambda,0} \rangle \neq 0$,

$$\ln Z_{N,\Lambda} = 2 \ln |\langle \Omega_\Lambda, \Omega_{\Lambda,0} \rangle| - t_0 E_\Lambda N + O(e^{-t_0 \gamma N}).$$

Conversely, if we show that for some constants a_1, a_2, a_3

$$\ln Z_{N,\Lambda} = a_1 + a_2 N + O(e^{-a_3 N}), \quad (8)$$

with $a_3 > 0$, this will imply that in the cyclic subspace generated by $\Omega_{\Lambda,0}$ the operator H_Λ has a gapped ground state. We will argue later that the asymptotic (8) holds, with the same a_2 and a_3 , if we add a small perturbation to $\Omega_{\Lambda,0}$ in (7), so H_Λ has a gapped ground state in the whole space \mathcal{H}_Λ . The non-degeneracy of the ground state can be deduced by a continuity argument from the non-degeneracy of the ground state in the non-perturbed system.

We begin proving (8) by writing the identity

$$e^{-t_0 H_\Lambda} = \sum_{I \subset \Lambda} T_{\Lambda,I},$$

where

$$T_{\Lambda,I} = \sum_{J \subset I} (-1)^{|I|-|J|} e^{-t_0 (H_{\Lambda,0} + \sum_{x \in J} \phi_x)}.$$

Here the operator $H_{\Lambda,J} \equiv H_{\Lambda,0} + \sum_{x \in J} \phi_x$ is defined by the KLMN theorem, like H_Λ . When formally Trotter or Duhamel expanded, $T_{\Lambda,I}$ is, by an inclusion-exclusion argument, the contribution to the total evolution from the perturbation of the classical evolution containing terms ϕ_x with $x \in I$ (see [17]). We do not explicitly use these expansions, however. Since all non-commutative terms ϕ_x in $T_{\Lambda,I}$ lie in $\Lambda_I \equiv \cup_{x \in I} (\Lambda_0 + x)$, we can write

$$T_{\Lambda,I} = T'_I e^{-t_0 H_{\Lambda \setminus \Lambda_I, 0}}, \quad (9)$$

where $H_{\Lambda \setminus \Lambda_I, 0} = \sum_{x \in \Lambda: (\Lambda_0 + x) \cap \Lambda_I = \emptyset} h_x$, and T'_I is defined as $T_{\Lambda,I}$ with $H_{\Lambda,0}$ replaced by $\sum_{x: (\Lambda_0 + x) \cap \Lambda_I \neq \emptyset} h_x = H_{\Lambda,0} - H_{\Lambda \setminus \Lambda_I, 0}$. For any $\Lambda_1 \subset \Lambda$ we will denote its neighborhood $\cup_{x: (\Lambda_0 + x) \cap \Lambda_1 \neq \emptyset} (\Lambda_0 + x)$ by $\tilde{\Lambda}_1$, so that T'_I acts on $\mathcal{H}_{\tilde{\Lambda}_I}$.

Lemma 1. $\|T'_I\| \leq (2\alpha e^{t_0 \beta / \alpha})^{|I|}$.

Proof. For some $J \subset I$, let $z_J \equiv (z_{x_1}, \dots, z_{x_{|J|}})$, $x_k \in J$, be a complex vector and consider the operator-valued function

$$H_J(z_J) = \sum_{x: (\Lambda_0 + x) \cap \Lambda_I \neq \emptyset} h_x + \sum_{x \in J} z_x \phi_x.$$

If all $|z_x| < 1/\alpha$, then, by (2), the quadratic form $\sum_{x \in J} z_x \phi_x$ is bounded relative to $\sum_{x: (\Lambda_0 + x) \cap \Lambda_I \neq \emptyset} h_x$, with a relative bound < 1 :

$$\left| \sum_{x \in J} z_x \phi_x(v, v) \right| \leq \max |z_x| \alpha \left\| \left(\sum_{x: (\Lambda_0 + x) \cap \Lambda_I \neq \emptyset} h_x \right)^{1/2} v \right\|^2 + \max |z_x| |J| \beta \|v\|^2. \quad (10)$$

Therefore $H_J(z_J)$ is an analytic family of m -sectorial operators on $\mathcal{H}_{\tilde{\Lambda}_I}$ for $z_J \in \{|z_x| < 1/\alpha | x \in J\}$ (see [14]). Since by (10) the numerical range $\{\langle H_J(z_J)v, v \rangle | v \in \text{Dom}(H_J(z_J)), \|v\| = 1\}$ of these operators lies in the half-plane $\{\text{Re } z \geq |J|\beta/\alpha\}$, it follows from the Hille-Yosida theorem that

$$\|e^{-t_0 H_J(z_J)}\| \leq e^{t_0 |J|\beta/\alpha}. \quad (11)$$

Now we consider the operator-valued function

$$T_I(z_I) = \sum_{J \subset I} (-1)^{|I|-|J|} e^{-t_0 H_J(z_J)},$$

where z_J is a restriction of z_I to J . The function $T_I(z_I)$ is analytic in $\{|z_x| < 1/\alpha | x \in J\}$ and, by (11), $\|T_I(z_I)\| \leq 2^{|I|} e^{t_0 |I|\beta/\alpha}$. Note that if $z_x = 0$ for some $x \in I$, then $T_I(z_I) = 0$ because in this case the terms $J \setminus \{x\}$ and $J \cup \{x\}$ make opposite contribution. Finally, T'_I appearing in (9) is the value of $T_I(z_I)$ at $z_I = (1, 1, \dots, 1)$. Now we use a many-dimensional version of the Schwarz lemma.

Lemma 2. *Let $f(z_I)$ be an analytic function in $\{|z_x| < a | x \in I\}$ and $|f(z_I)| \leq M$ for all z_I . Suppose that if $z_x = 0$ for some x , then $f(z_I) = 0$. Then $|f(z_I)| \leq M a^{-|I|} \prod_{x \in I} |z_x|$.*

This lemma follows by induction from the usual one-dimensional Schwarz lemma. Applying it to $T_I(z_I)$, we obtain the desired estimate. \square

By expanding $e^{-t_0 H_\Lambda}$ in $T_{\Lambda, I}$ we have isolated the regions with non-classical evolution; to obtain the final cluster expansion we need to isolate in addition regions with classically evolving excited states. Denote $\Lambda_I = \cup_{x \in I} (\Lambda_0 + x)$, and also

$$\mathcal{H}'_x \equiv \mathcal{H}_x \ominus \Omega_x, \quad \mathcal{H}'_{\Lambda_1} \equiv \otimes_{x \in \Lambda_1} \mathcal{H}'_x,$$

and write $T_{\Lambda, I}$ as

$$T_{\Lambda, I} = T_{\Lambda, I} \sum_{J \subset \Lambda \setminus \Lambda_I} P_{\mathcal{H}'_J} P_{\Omega_{(\Lambda \setminus \Lambda_I) \setminus J, 0}}.$$

Now define a configuration C as a sequence $\{(I_k, J_k) | k = 1, \dots, N\}$, where $J_k \subset \Lambda \setminus \Lambda_{I_k}$; it follows that

$$Z_{N, \Lambda} = \sum_C w(C),$$

where

$$w(C) = \left\langle \prod_{k=1}^N \left(T_{\Lambda, I_k} P_{\mathcal{H}'_{J_k}} P_{\Omega_{(\Lambda \setminus \Lambda_{I_k}) \setminus J_k, 0}} \right) \Omega_{\Lambda, 0}, \Omega_{\Lambda, 0} \right\rangle \quad (12)$$

with the time-ordered product $\prod_{k=1}^N A_k \equiv A_N \cdots A_1$.

Lemma 3. $|w(C)| \leq \prod_{k=1}^N \left((2\alpha e^{t_0\beta/\alpha})^{|I_k|} e^{-t_0(|J_k| - |\Lambda_0|^3|I_k|)} \right).$

Proof. We estimate the norm of the operator in round brackets in (12). By Lemma 1, $\|T'_I\| \leq (2\alpha e^{t_0\beta/\alpha})^{|I|}$. Next, if $J \subset \Lambda \setminus \Lambda_I$, then $|J \cap (\Lambda \setminus \tilde{\Lambda}_I)| \geq |J| - |\Lambda_0|^2|I| \geq |J| - |\Lambda_0|^3|I|$. Any $x \in \Lambda \setminus \tilde{\Lambda}_I$ belongs to $|\Lambda_0|$ sets of the form $\Lambda_0 + y$, these sets don't overlap with Λ_I and all contain $|\Lambda_0|$ sites; therefore by the spectral gap assumption about h_x

$$H_{\Lambda \setminus \Lambda_I, 0} \Big|_{\mathcal{H}'_J \otimes \Omega_{(\Lambda \setminus \Lambda_I) \setminus J, 0}} \geq |J \cap (\Lambda \setminus \tilde{\Lambda}_I)| \mathbf{1} \geq (|J| - |\Lambda_0|^3|I|) \mathbf{1}.$$

It follows that the norm of the expression in brackets in (12) does not exceed $(2\alpha e^{t_0\beta/\alpha})^{|I_k|} e^{-t_0(|J_k| - |\Lambda_0|^3|I_k|)}$, which implies the desired estimate. \square

Now for a configuration C we define its support $\text{supp } C \subset \{0, 1, \dots, N\} \times \Lambda$ as the set

$$\{(k, x) | k = 0, \dots, N; x \in \tilde{\Lambda}_{I_k} \cup \tilde{\Lambda}_{I_{k+1}} \cup \tilde{J}_k \cup \tilde{J}_{k+1}\} \quad (13)$$

(with $\tilde{\Lambda}_{I_0} = \tilde{\Lambda}_{I_{N+1}} = \tilde{J}_0 = \tilde{J}_{N+1} = \emptyset$). We say that configurations are disjoint if they have disjoint supports. If C_1 and C_2 are disjoint, we naturally define their union $C = C_1 \cup C_2$ as the configuration with $I_k = I_k^{(1)} \cup I_k^{(2)}$, $J_k = I_k^{(1)} \cup J_k^{(2)}$.

Lemma 4. *If C_1 and C_2 are disjoint, then $w(C_1 \cup C_2) = w(C_1)w(C_2)$.*

Proof. For the configuration $C = C_1 \cup C_2$ and any $n = 1, \dots, N$ consider the vector

$$v_n = \prod_{k=1}^n \left(T_{\Lambda, I_k} P_{\mathcal{H}'_{J_k}} P_{\Omega_{(\Lambda \setminus \Lambda_{I_k}) \setminus J_k, 0}} \right) \Omega_{\Lambda, 0},$$

so that $w(C) = \langle v_N, \Omega_{\Lambda, 0} \rangle$. We have $v_n = u_n \otimes \Omega_{\Lambda \setminus (\Lambda_{I_n} \cup J_n), 0}$ with some $u_n \in \mathcal{H}_{\Lambda_{I_n} \cup J_n}$. Analogously, we can define $v_n^{(1)}, v_n^{(2)}, u_n^{(1)}, u_n^{(2)}$ for C_1, C_2 . Let $K_n = \Lambda_{I_n} \cup J_n$ and similarly define $K_n^{(1)}, K_n^{(2)}$ for C_1, C_2 . Since $\text{supp } C_1$ and $\text{supp } C_2$ are disjoint, it follows in particular that $K_n^{(1)}$ and $K_n^{(2)}$ are disjoint, so that $\mathcal{H}_{K_n} = \mathcal{H}_{K_n^{(1)}} \otimes \mathcal{H}_{K_n^{(2)}}$. We will prove by induction that $u_n = u_n^{(1)} \otimes u_n^{(2)}$; at $n = N$ this implies the desired equality $w(C_1 \cup C_2) = w(C_1)w(C_2)$. Suppose that $u_{n-1} = u_{n-1}^{(1)} \otimes u_{n-1}^{(2)}$. Note that we have in $\mathcal{H}_{\tilde{K}_n \cup K_{n-1}}$ the equality

$$u_n \otimes \Omega_{(\tilde{K}_n \cup K_{n-1}) \setminus K_n, 0} = T'_{K_n} P_{\Omega_{(\tilde{K}_n \cup K_{n-1}) \setminus K_n, 0}} (u_{n-1} \otimes \Omega_{(\tilde{K}_n \cup K_{n-1}) \setminus K_{n-1}, 0}), \quad (14)$$

where T'_{K_n} is defined as $T_{\Lambda, I}$ with $H_{\Lambda, 0}$ replaced by $\sum_{x: (\Lambda_0 + x) \cap K_n \neq \emptyset} h_x = H_{\Lambda, 0} - H_{\Lambda \setminus K_n, 0}$. (14) holds because $e^{-t_0(H_{\Lambda, 0} - H_{\Lambda \setminus K_n, 0})}$ acts trivially on the ground state. By the disjointness, the objects in (14) factor into products of respective objects for C_1, C_2 , which proves the inductive step. \square

A polymer χ is a connected configuration (i.e., which is not a union of two configurations with disjoint supports). We have

$$Z_{N,\Lambda} = \sum_{\text{disj. } \chi_1, \dots, \chi_n}^{N,\Lambda} \prod_{k=1}^n w(\chi_k),$$

where summation is over all disjoint collections of polymers in $\{1, \dots, N\} \times \Lambda$. This is the desired polymer expansion. By Lemma 3, for any $\epsilon > 0$ we can choose t_0 large and then α, β small so that $w(\chi) \leq \epsilon^{|\text{supp } \chi|}$. A standard combinatorial argument shows that the number of polymers with $|\text{supp } \chi| = n$ containing a given point does not exceed c^n for some $c = c(\nu, \Lambda_0)$. Now all conclusions of Theorem 1 follow from standard results on cluster expansions [20, 21, 26, 27, 17], and we will be very sketchy. We define a cluster X as a connected collection of polymers χ_1, \dots, χ_k with positive multiplicities n_1, \dots, n_k . Let $G(X)$ be a graph with $n_1 + \dots + n_k$ vertices, corresponding to these polymers, and a line between two vertices drawn if the corresponding polymers intersect. Let $G_1 \triangleleft G(X)$ stand for a connected subgraph G_1 containing all vertices of $G(X)$, and $l(G_1)$ be the number of lines in G_1 . Then the weight of the cluster X is defined as

$$w(X) = (n_1! \cdots n_k!)^{-1} w(\chi_1)^{n_1} \cdots w(\chi_k)^{n_k} \sum_{G_1 \triangleleft G(X)} (-1)^{l(G_1)}.$$

It follows that

$$\ln \sum_{\text{disj. } \chi_1, \dots, \chi_n}^{N,\Lambda} \prod_{k=1}^n w(\chi_k) = \sum_X^{N,\Lambda} w(X),$$

with the absolutely convergent series on the r.h.s. (see [6, 28] for recent simple proofs). Let $l(X)$ be the time length of a cluster; shifting clusters in time, we write

$$\begin{aligned} \sum_X^{N,\Lambda} w(X) &= \sum_{X: l(X) \leq N}^{t=0,\Lambda} (N - l(X)) w(X) \\ &= - \sum_X^{t=0,\Lambda} l(X) w(X) + N \sum_X^{t=0,\Lambda} w(X) + \sum_{X: l(X) > N}^{t=0,\Lambda} (l(X) - N) w(X), \end{aligned}$$

where $\sum^{t=0,\Lambda}$ is the sum over clusters starting at $t = 0$. By the cluster estimate, all series in the r.h.s. converge absolutely, and the last term is $O(\epsilon^N)$ because summation is over clusters with length $> N$. Comparing this with (8), we identify a_1 as $-\sum_X^{t=0,\Lambda} l(X) w(X)$, a_2 as $\sum_X^{t=0,\Lambda} w(X)$, and a_3 as $-\ln \epsilon$. This ϵ does not depend on Λ , so the spectral gap estimate is volume-independent. To complete the proof of 1), we consider the changes in the

asymptotic (8) when $\Omega_{\Lambda,0}$ is replaced by $\Omega_{\Lambda,0} + v$ with small v . This replacement adds new polymers χ_v , arising from the new terms $\langle (e^{-t_0 H_\Lambda})^N v, \Omega_{\Lambda,0} \rangle$, $\langle (e^{-t_0 H_\Lambda})^N \Omega_{\Lambda,0}, v \rangle$ and $\langle (e^{-t_0 H_\Lambda})^N v, v \rangle$. The support of χ_v contains $\{0\} \times \Lambda$ or $\{N\} \times \Lambda$, or both. For v small enough the estimate $|w(\chi)| \leq \epsilon^{|\text{supp } \chi|}$ remains valid for χ_v . The expansion for $\ln Z_{N,\Lambda}$ is modified by adding clusters containing the new polymers χ_v . Such clusters touch the boundary of the time segment $\{0, \dots, N\}$, and hence their contribution is $c + O(\epsilon^N)$. This completes the proof of 1). To show that 2) holds, one writes

$$\langle A\Omega_\Lambda, \Omega_\Lambda \rangle = \lim_{N \rightarrow \infty} Z_{2N,\Lambda}^{-1} \langle A e^{-t_0 N H_\Lambda} \Omega_{\Lambda,0}, e^{-t_0 N H_\Lambda} \Omega_{\Lambda,0} \rangle,$$

using the fact that $\langle \Omega_{\Lambda,0}, \Omega_\Lambda \rangle \neq 0$. If A acts on \mathcal{H}_{Λ_A} , one introduces new polymers χ_A with the support containing $\{0\} \times \Lambda_A$ and the weight calculated using A inserted in the 0th layer; one has $|w(\chi_A)| \leq \epsilon^{|\text{supp } \chi_A| - |\Lambda_A|} \|A\|$. It follows that $\langle A\Omega_\Lambda, \Omega_\Lambda \rangle = \sum_X^\Lambda w(X)$, where the sum is over clusters in $\mathbb{Z} \times \Lambda$, containing one polymer χ_A with multiplicity 1. As $\Lambda \nearrow \mathbb{Z}^\nu$, this expression tends to the absolutely convergent sum over polymers in $\mathbb{Z} \times \mathbb{Z}^\nu$, which proves 2). 3) follows from the fact that the truncated correlation on the l.h.s. equals $\sum_X w(X)$ over clusters containing either a polymer with the support containing $\{0\} \times \Lambda_1$ and $\{0\} \times \Lambda_2$, or two polymers χ_{A_1}, χ_{A_2} . Finally, 4) follows because if ϕ varies analytically, then the cluster expansion does too and is convergent as long as the estimate (2) holds.

3 Proof of Theorem 2

Following [17], in order to extend the perturbation theory to α close to 1 we use a scaling transformation. We group lattice sites in cubic blocks b_x of linear size l , so that the initial cubic volume Λ is transformed into cubic volume $\bar{\Lambda}$ whose sites x are these blocks (we assume that the size of Λ is a multiple of l , but one can consider general cubic volumes too by taking blocks of different sizes). For any $x \in \bar{\Lambda}$, let

$$\bar{\mathcal{H}}_x = \otimes_{y \in b_x} \mathcal{H}_y, \quad \bar{\Omega}_x = \otimes_{y \in b_x} \Omega_y, \quad \bar{\mathcal{H}}'_x = \bar{\mathcal{H}}_x \ominus \bar{\Omega}_x$$

and also

$$\bar{\mathcal{H}}'_I = \otimes_{x \in I} \bar{\mathcal{H}}'_x, \quad \bar{\Omega}_{I,0} = \otimes_{x \in I} \bar{\Omega}_x$$

for $I \subset \bar{\Lambda}$. Suppose that $l > \text{diam}(\Lambda_0)$. We can then view the interaction Φ_Λ as the sum $\sum_{x \in \bar{\Lambda}} \bar{\phi}_x$; here $\bar{\phi}_x$ acts on $\bar{\mathcal{H}}_{\bar{\Lambda}_0+x}$, where $\bar{\Lambda}_0 = \{0, 1\}^\nu$, and is defined as

$$\bar{\phi}_x = \sum_{y: \Lambda_0+y \in \cup_{z \in \bar{\Lambda}_0+x} b_z} c_y^{-1} \phi_y. \quad (15)$$

Here $c_y = |\{x : \Lambda_0 + y \in \cup_{z \in \bar{\Lambda}_0 + x} b_z\}|$ is the number of $\bar{\phi}_x$ where ϕ_y appears. We define analogously $\bar{\phi}_x^{(r)}, \bar{\phi}_x^{(b)}$ for $\phi_x^{(r)}, \phi_x^{(b)}$ appearing in the decomposition (3). Next for any $I \subset \bar{\Lambda}$ we define

$$\bar{\Phi}_I = \sum_{x \in I} \bar{\phi}_x$$

and similarly for $\bar{\Phi}_I^{(r)}, \bar{\Phi}_I^{(b)}$, so that $H_\Lambda = H_{\Lambda,0} + \bar{\Phi}_\Lambda^{(r)} + \bar{\Phi}_\Lambda^{(b)}$. Following the proof of Theorem 1, we write

$$e^{-t_0 H_\Lambda} = \sum_{I, J, K \subset \bar{\Lambda}} T_{I, J, K},$$

where

$$T_{I, J, K} = \sum_{I_1 \subset I} \sum_{J_1 \subset J} (-1)^{|I| - |I_1|} (-1)^{|J| - |J_1|} e^{-t_0 (H_{\Lambda,0} + \bar{\Phi}_{I_1}^{(r)} + \bar{\Phi}_{J_1}^{(b)})} P_{\bar{\mathcal{H}}'_K \otimes \bar{\Omega}_{\bar{\Lambda} \setminus K, 0}}.$$

We call a sequence $C = \{(I_k, J_k, K_k), k = 1, \dots, N\}$ a configuration and assign to it the weight $w(C) = \langle \prod_{k=1}^N \bar{T}_{I_k, J_k, K_k} \bar{\Omega}_{\bar{\Lambda}, 0}, \bar{\Omega}_{\bar{\Lambda}, 0} \rangle$. Let x_1, \dots, x_ν stand for the coordinates of the site $x \in \bar{\Lambda}$; for any set $I \subset \bar{\Lambda}$ we define its neighborhood

$$\tilde{I} = \{x | \exists y \in I \text{ such that } |x_k - y_k| \leq 1, k = 1, \dots, \nu\}.$$

Let $\bar{\Lambda}_I = \cup_{x \in I} (\bar{\Lambda}_0 + x)$. Similarly to (13), we define a configuration's support as

$$\{(k, x) | k = 0, \dots, N; x \in \tilde{\bar{\Lambda}}_{I_k} \cup \tilde{\bar{\Lambda}}_{J_{k+1}} \cup \tilde{\bar{\Lambda}}_{J_k} \cup \tilde{\bar{\Lambda}}_{J_{k+1}} \cup \tilde{K}_k \cup \tilde{K}_{k+1}\}.$$

An analog of Lemma 4 on factorization of weights is immediate. Therefore the only thing that needs to be proved is an exponential bound for the weight:

$$|w(C)| \leq \epsilon^{|\text{supp } C|}$$

with sufficiently small ϵ ; after that the conclusion of the theorem follows like in the previous section. Clearly, in order to have this bound it suffices to show that for any ϵ one can choose t_0, l and β so that

$$\|T_{I, J, K}\| \leq \epsilon^{|I| + |J| + |K|}. \quad (16)$$

The remainder of this section is a proof of this claim. Specifically, we will show that for α close to 1 one can achieve this by choosing

$$t_0 = (1 - \alpha)^{-\varkappa}, \quad l = \lceil (1 - \alpha)^{-\varkappa} \rceil, \quad \beta = \delta(1 - \alpha)^{\varkappa(\nu+1)}, \quad (17)$$

where $\lceil \cdot \rceil$ is the integer part; here \varkappa is any fixed constant > 1 , and $\delta = \delta(\varkappa, \nu, \Lambda_0)$ a sufficiently small constant. The strategy of the proof is as follows. We will obtain three different bounds for $\|T_{I,J,K}\|$, suitable when the contribution to $I \cup J \cup K$ from either I, J or K is large enough.

Case 1. The first bound relies on the smallness of the bounded part $\overline{\phi}^{(b)}$ of the perturbation, and is used when J is large. In this case we use again the Schwarz lemma. In the definition of $T_{I,J,K}$ we replace $\sum_{x \in J_1} \overline{\phi}_x^{(b)}$ with $\sum_{x \in J_1} z_x \overline{\phi}_x^{(b)}$, where $z_x \in \mathbb{C}$, $|z_x| \leq a$, with some $a > 1$. Let $a = (t_0 \|\overline{\phi}^{(b)}\|)^{-1}$, then by definition of $\overline{\phi}^{(b)}$ and from (17)

$$a \geq (t_0(2l)^\nu \|\overline{\phi}^{(b)}\|)^{-1} \geq (t_0(2l)^\nu \beta)^{-1} \geq 2^{-\nu} \delta^{-1} > 1$$

if $\delta < 2^{-\nu}$. Using the Schwarz lemma with this a , we find that

$$\|T_{I,J,K}\| \leq 2^{|I|} (2et_0 \|\overline{\phi}^{(b)}\|)^{|J|} \leq 2^{|I|} (2^{\nu+1} e \delta)^{|J|}. \quad (18)$$

Case 2. The second bound relies on the contractiveness of the classical evolution in excited regions and is used when K is large. We will estimate the norm of $e^{-t_0(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)})} P_{\overline{\mathcal{H}}_K \otimes \overline{\Omega}_{\Lambda \setminus K,0}}$. We begin by writing

$$e^{-t_0(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)})} = (2\pi i)^{-1} \int_{\Gamma} e^{-t_0 z} R_z dz, \quad (19)$$

where R_z is the resolvent $(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)} - z)^{-1}$, and Γ is a contour in the complex plane going around the spectrum of $H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)}$; we will specify Γ below. We will use the expansion of the resolvent

$$R_z = Q_z \left(\sum_{k=0}^{\infty} F_z^k \right) Q_z, \quad (20)$$

where

$$F_z = -(H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z)^{-1/2} \overline{\Phi}_{I_1}^{(r)} (H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z)^{-1/2}$$

and

$$Q_z = (H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z)^{-1/2}.$$

The operators Q_z are well-defined and bounded for z in the resolvent set of $H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)}$, i.e., at least in $\mathbb{C} \setminus [-\|\overline{\Phi}_{J_1}^{(b)}\|, +\infty)$. We now estimate the norm of F_z :

$$\begin{aligned}
\|F_z\| &= \| |H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{-1/2} \overline{\Phi}_{I_1}^{(r)} | H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{-1/2} \| \\
&= \sup_{u \in \overline{\mathcal{H}}_{\Lambda} \setminus \{0\}} \frac{|\langle |H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{-1/2} \overline{\Phi}_{I_1}^{(r)} | H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{-1/2} u, u \rangle|}{\|u\|^2} \\
&= \sup_{v \in \text{Dom}(|H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{1/2})} \frac{|\overline{\Phi}_{I_1}^{(r)}(v, v)|}{\| |H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{1/2} v \|^2} \\
&\leq \sup_{v \in \text{Dom}(|H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{1/2})} \frac{\alpha \|H_{\Lambda,0}^{1/2} v\|^2}{\| |H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)} - z|^{1/2} v \|^2} \\
&\leq \sup_{\lambda \in \text{Spec}(H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)})} \frac{\alpha(\lambda + \|\overline{\Phi}_{J_1}^{(b)}\|)}{|\lambda - z|}.
\end{aligned}$$

We will be interested in those z where $\|F_z\| \leq \sqrt{\alpha}$. By the above bound, a sufficient condition for that is

$$z \notin \bigcup_{\lambda \in \text{Spec}(H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)})} \left\{ z \in \mathbb{C} \mid |z - \lambda| \leq \sqrt{\alpha}(\lambda + \|\overline{\Phi}_{J_1}^{(b)}\|) \right\}. \quad (21)$$

Since $\text{Spec}(H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)}) \subset [-\|\overline{\Phi}_{J_1}^{(b)}\|, +\infty)$, the above union lies in the sector

$$\{z \in \mathbb{C} : |\arg(z + \|\overline{\Phi}_{J_1}^{(b)}\|)| \leq \arcsin \sqrt{\alpha}\}, \quad (22)$$

so if we choose z outside this sector we have $\|F_z\| \leq \sqrt{\alpha}$ and in particular

$$\|Q_z F_z^k Q_z\| \leq \frac{\alpha^{k/2}}{\text{dist}(z, \text{Spec}(H_{\Lambda,0} + \overline{\Phi}_{J_1}^{(b)}))}. \quad (23)$$

We will now show that, furthermore, one can enlarge the domain of z where a bound of the above type holds, if one applies the operator on the l.h.s. to vectors from $\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}$ with K large compared to J . Precisely, let

$$n = \lceil (|K| - 6^\nu |J|) / 7^\nu \rceil.$$

This n is a lower bound for the maximal number of sites in a subset $K_1 \subset K$ such that the neighborhoods $\widetilde{\{x\}}$ of points $x \in K_1$ are separated from each other and from $\overline{\Lambda}_J$ by at least two $\overline{\Lambda}$ -lattice spacings: choose the first such x outside the 2-neighborhood of $\overline{\Lambda}_J$, then the second outside the 2-neighborhood of $\overline{\Lambda}_J$ unioned with the 3-neighborhood of the first x , etc. We assume that $n > 0$. Next, let

$$m = \lceil l / \text{diam}(\Lambda_0) \rceil, m > 0.$$

For any a define \mathcal{U}_a as the union of circles standing in (21), but with λ running over $[a, +\infty)$. Now, suppose that $\lceil k/m \rceil = r$ with $r \leq n$; we claim then that

$$\|Q_z F_z^k Q_z P_{\overline{\mathcal{H}}_K \otimes \overline{\Omega}_{\Lambda \setminus K, 0}}\| \leq \frac{\alpha^{k/2}}{\text{dist}(z, [\Lambda_0|(n-r) - \|\overline{\Phi}_{J_1}^{(b)}\|, +\infty))}, \quad (24)$$

$$\forall z \notin \mathcal{U}_{|\Lambda_0|(n-r) - \|\overline{\Phi}_{J_1}^{(b)}\|}.$$

Indeed, let $\mathcal{G}_{[a, +\infty)}$ stand for the spectral subspace of $H_{\Lambda, 0} + \overline{\Phi}_{J_1}^{(b)}$, corresponding to the interval $[a, +\infty)$. Let K_1 be chosen as above, $|K_1| = n$. Then the subspace $\overline{\mathcal{H}}'_{K_1} \otimes \overline{\mathcal{H}}_{\Lambda \setminus K_1}$ is an invariant subspace of $H_{\Lambda, 0} + \overline{\Phi}_{J_1}^{(b)}$ and it lies in $\mathcal{G}_{[\Lambda_0|n - \|\overline{\Phi}_{J_1}^{(b)}\|, +\infty)}$. The sites $x \in K_1$ (and therefore their neighborhoods $\widetilde{\{x\}}$ too) are in excited states. When we apply powers of F_z to vectors from $\overline{\mathcal{H}}'_{K_1} \otimes \overline{\mathcal{H}}_{\Lambda \setminus K_1}$, we need at least $m = \lceil l/\text{diam}(\Lambda_0) \rceil$ powers to remove the excitation from a neighborhood $\widetilde{\{x\}}$ of a given $x \in K_1$, because the block b_x has to be connected to $\Lambda \setminus \bigcup_{y \in \widetilde{\{x\}}} b_y$ by supports of the elementary interactions $\phi_w^{(r)}$, $w \in \Lambda$, of which $\overline{\Phi}_{I_1}^{(r)}$ is composed. Hence to remove excitations from r neighborhoods we need at least mr powers of F_z . Therefore if $k < mr$ with $r \leq n$, then

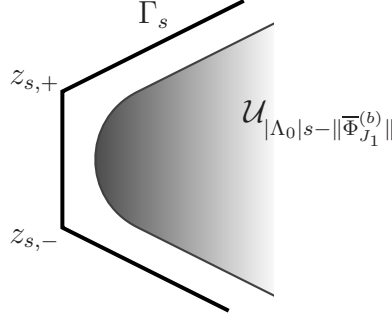
$$F_z^k \overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\Lambda \setminus K, 0} \subset \mathcal{G}_{[\Lambda_0|(n-r) - \|\overline{\Phi}_{J_1}^{(b)}\|, +\infty)}$$

(here we use $\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\Lambda \setminus K, 0} \subset \overline{\mathcal{H}}'_{K_1} \otimes \overline{\mathcal{H}}_{\Lambda \setminus K_1}$). It follows that in the case at hand we can replace $H_{\Lambda, 0} + \overline{\Phi}_{J_1}^{(b)}$ and the quadratic form $\overline{\Phi}_{I_1}^{(r)}$ with their restrictions to $\mathcal{G}_{[\Lambda_0|(n-r) - \|\overline{\Phi}_{J_1}^{(b)}\|, +\infty)}$; the argument leading to (23) then yields (24).

Now we can specify the integration contour Γ . It will depend on the term k in the expansion (20) through $r = \lceil k/m \rceil$. Let $s = n - r$. For $s = 0, \dots, n$, let

$$z_{s, \pm} = (1 - \sqrt{\alpha})|\Lambda_0|s - \|\overline{\Phi}_{J_1}^{(b)}\| \pm i \frac{\sqrt{\alpha}}{\sqrt{1 - \alpha}}|\Lambda_0|s - t_0^{-1}$$

and let the contour Γ_s consist of the segment $[z_{s, -}, z_{s, +}]$ and the two rays $\{z \mid \arg(z - z_{s, \pm}) = \pm \arcsin \sqrt{\alpha}\}$, so that it is the boundary of the truncated sector (22) (of the sector itself in the case $s = 0$) shifted by t_0^{-1} to the left. This contour is just a convenient for calculations, piecewise-linear approximation of the boundary of $\mathcal{U}_{|\Lambda_0|s - \|\overline{\Phi}_{J_1}^{(b)}\|}$. We choose the contour in this way because we need it to lie as far to the right as possible due to the factor $e^{-t_0 z}$ in (19), but still outside of $\mathcal{U}_{|\Lambda_0|s - \|\overline{\Phi}_{J_1}^{(b)}\|}$, so that the resolvent estimate can be used. By slightly shifting it to the left we avoid the possible singularity in the denominator in (24).



We have $\Gamma_s \cap \mathcal{U}_{|\Lambda_0|s - \|\bar{\Phi}_{J_1}^{(b)}\|} = \emptyset$, so we can write

$$\begin{aligned}
& e^{-t_0(H_{\Lambda,0} + \bar{\Phi}_{I_1}^{(r)} + \bar{\Phi}_{J_1}^{(b)})} P_{\mathcal{H}'_K \otimes \bar{\Omega}_{\Lambda \setminus K,0}} \\
&= (2\pi i)^{-1} \sum_{r=0}^{n-1} \int_{\Gamma_{n-r}} e^{-t_0 z} Q_z \left(\sum_{k=r}^{(r+1)m-1} F_z^k \right) Q_z P_{\mathcal{H}'_K \otimes \bar{\Omega}_{\Lambda \setminus K,0}} dz \\
&+ (2\pi i)^{-1} \int_{\Gamma_0} e^{-t_0 z} Q_z \left(\sum_{k=n}^{\infty} F_z^k \right) Q_z P_{\mathcal{H}'_K \otimes \bar{\Omega}_{\Lambda \setminus K,0}} dz.
\end{aligned}$$

We estimate now this expression using $|\int e^{-t_0 z} f(z) dz| \leq \int e^{-t_0 \operatorname{Re} z} |f(z)| |dz|$ and the bound (24). We have $\operatorname{dist}(\Gamma_s, [|\Lambda_0|s - \|\bar{\Phi}_{J_1}^{(b)}\|, +\infty)) \geq \sqrt{\alpha}$. It follows that

$$\begin{aligned}
& \|e^{-t_0(H_{\Lambda,0} + \bar{\Phi}_{I_1}^{(r)} + \bar{\Phi}_{J_1}^{(b)})} P_{\mathcal{H}'_K \otimes \bar{\Omega}_{\Lambda \setminus K,0}}\| \\
&\leq \frac{e^{t_0 \|\bar{\Phi}_{J_1}^{(b)}\| + 1}}{2\pi \sqrt{1-\alpha}(1-\sqrt{\alpha})} \left(\sum_{r=0}^{n-1} \left[(\alpha^{\frac{rm}{2}} - \alpha^{\frac{(r+1)m}{2}}) e^{-t_0(1-\sqrt{\alpha})|\Lambda_0|(n-r)} (|\Lambda_0|(n-r) \right. \right. \\
&\quad \left. \left. + \frac{2}{\sqrt{\alpha t_0}}) \right] + \frac{2\alpha^{\frac{nm}{2}}}{\sqrt{\alpha t_0}} \right) \\
&\leq \frac{e^{t_0 \|\bar{\Phi}_{J_1}^{(b)}\| + 1}}{2\pi \sqrt{1-\alpha}} (n+1) \left(|\Lambda_0|n + \frac{2}{\sqrt{\alpha t_0}} \right) (\max\{\alpha^{\frac{m}{2}}, e^{-t_0(1-\sqrt{\alpha})|\Lambda_0|}\})^n.
\end{aligned}$$

Some calculation now shows that if t_0, l, β are defined as in (17) with $\varkappa > 1$, then for any $\epsilon > 0$ we have

$$\|e^{-t_0(H_{\Lambda,0} + \bar{\Phi}_{I_1}^{(r)} + \bar{\Phi}_{J_1}^{(b)})} P_{\mathcal{H}'_K \otimes \bar{\Omega}_{\Lambda \setminus K,0}}\| \leq e^{t_0 \|\bar{\Phi}_{J_1}^{(b)}\|} \epsilon^n \leq e^{2^\nu \delta |J_1|} \epsilon^n$$

when $\alpha > \alpha_0$ for some $\alpha_0 = \alpha_0(\epsilon, \varkappa, \nu, \Lambda_0) < 1$. It follows that

$$\|T_{I,J,K}\| \leq 2^{|I|+|J|} e^{2^\nu \delta |J|} \epsilon^n. \tag{25}$$

Case 3. The third bound is used when I is large. We will bound $\|\sum_{I_1 \subset I} (-1)^{|I|-|I_1|} e^{-t_0(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)})} P_{\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}}\|$. Like in case 2, we represent

$$e^{-t_0(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)})} = (2\pi i)^{-1} \int_{\Gamma_0} e^{-t_0 z} R_{z, I_1} dz,$$

where $R_{z, I_1} = (H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)} - z)^{-1}$ and Γ_0 is the shifted boundary of the sector defined in case 2. We again expand $R_{z, I_1} = Q_z (\sum_{k=0}^{\infty} F_{z, I_1}^k) Q_z$. Now, let

$$n' = \lceil (|I| - 6^\nu |J| - 5^\nu |K|) / 6^\nu \rceil.$$

This n' is a lower bound for the maximal number of sites in a subset $I_2 \subset I$ such that the neighborhoods $\{\widetilde{\Lambda_0 + x}\}$ of points $x \in I_2$ are separated from each other, from $\overline{\Lambda}_J$ and from K . Let $m = \lceil l / \text{diam}(\Lambda_0) \rceil$ as before. We claim that

$$\sum_{I_1 \subset I} (-1)^{|I|-|I_1|} F_{z, I_1}^k Q_z P_{\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}} = 0$$

if $k < mn'$. Indeed, if in the above sum we expand $F_{z, I_1} = -Q_z \sum_{x \in I_1} \overline{\phi}_x^{(r)} Q_z$, then by the inclusion-exclusion principle it becomes

$$(-1)^k \sum \left(\prod_{p=1}^k (Q_z \overline{\phi}_{x_p}^{(r)} Q_z) \right) Q_z P_{\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}},$$

where summation is over sequences x_1, \dots, x_k of points from I , containing each point of I at least once. In particular, each sequence contains all points of I_2 defined above. Each term in this sum is 0. Indeed, if we further expand each $\overline{\phi}_x^{(r)}$ in $\phi_y^{(r)}$ as in (15) and consider the set $\cup_{p=1}^k (\Lambda_0 + y_p)$ for each of the resulting sequences, then, if $k < mn'$, this set will have at least one connected component contained in the set $\{\widetilde{\Lambda_0 + x}\}$ for some $x \in I_2$. By the choice of I_2 , the operator $\phi_y^{(r)}$ from this component which acts first acts on the ground state. Since $\phi_y^{(r)} \Omega_{\Lambda_0 + y} = 0$, this implies our claim.

It follows that

$$\begin{aligned} & \left\| \sum_{I_1 \subset I} (-1)^{|I|-|I_1|} e^{-t_0(H_{\Lambda,0} + \overline{\Phi}_{I_1}^{(r)} + \overline{\Phi}_{J_1}^{(b)})} P_{\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}} \right\| \\ &= \left\| \sum_{I_1 \subset I} (-1)^{|I|-|I_1|} (2\pi i)^{-1} \int_{\Gamma_0} e^{-t_0 z} Q_z \sum_{k=mn'}^{\infty} F_{z, I_1}^k Q_z P_{\overline{\mathcal{H}}'_K \otimes \overline{\Omega}_{\overline{\Lambda} \setminus K, 0}} dz \right\| \\ &\leq \frac{2^{|I|} e^{t_0 \|\overline{\Phi}_{J_1}^{(b)}\| + 1} \alpha^{\frac{mn'}{2}}}{\pi t_0 \sqrt{\alpha(1-\alpha)} (1 - \sqrt{\alpha})}. \end{aligned}$$

Again, for any $\epsilon > 0$, if α is sufficiently close to 1, then this expression does not exceed $2^{|I|}e^{2^\nu\delta|J_1|}\epsilon^{n'}$, which implies

$$\|T_{I,J,K}\| \leq 2^{|I|+|J|}e^{2^\nu\delta|J|}\epsilon^{n'} \quad (26)$$

in case 3.

The desired bound (16) now follows from bounds obtained in the three cases. If I is sufficiently large compared to J and K , then one uses the bound (26). Otherwise, and if K is sufficiently large compared to J , one uses (25). In the remaining case one uses (18).

4 Proof of Theorem 3

We summarize some known facts about the AKLT model which we will need. Denote by H_Λ^p and H_Λ^f the AKLT Hamiltonians on a finite chain Λ with periodic and free boundary conditions, respectively. The Hamiltonian H_Λ^f has a four-dimensional subspace \mathcal{G}_Λ of frustration-free ground states. Using the valence-bond-solid representation, one can choose a (non-orthogonal) basis $\Omega_{\Lambda;ab}$, $a, b = 1, 2$, in \mathcal{G}_Λ with the following properties:

- 1) For two adjacent finite chains Λ_1, Λ_2

$$\Omega_{\Lambda_1 \cup \Lambda_2; ab} = \Omega_{\Lambda_1; a1} \otimes \Omega_{\Lambda_2; 2b} - \Omega_{\Lambda_1; a2} \otimes \Omega_{\Lambda_2; 1b}.$$

The unique ground state of H_Λ^p is given by $\Omega_{\Lambda;12} - \Omega_{\Lambda;21}$.

- 2) Let g_Λ be the 4×4 Gram matrix of the basis $\Omega_{\Lambda;ab}$:

$$(g_\Lambda)_{ab,cd} = \langle \Omega_{\Lambda;ab}, \Omega_{\Lambda;cd} \rangle.$$

Then $g_\Lambda = \mathbf{1} + O(3^{-|\Lambda|})$ as $|\Lambda| \rightarrow \infty$.

We will also use the fact that the operators H_Λ^f have a uniformly bounded away from 0 spectral gap, i.e. for some $\gamma > 0$ we have $H_\Lambda^f \geq \gamma(\mathbf{1} - G_\Lambda)$ for all Λ , where G_Λ is the projector onto \mathcal{G}_Λ .

All the above facts were proved in [3], see also [5, 9, 11, 15, 17, 19, 24] for various refinements.

We show now that at large scale the AKLT model is a perturbation of a non-interacting model. Like in the previous section, we group the sites of the cyclic chain Λ in blocks $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ with $|\Lambda_k| = l, k = 1, \dots, n$. We will specify l later. We write

$$H_\Lambda^p = \sum_{k=1}^n \overline{H}_{k,k+1},$$

where

$$\begin{aligned}\overline{H}_{k,k+1} &= H_{\Lambda_k}^f/2 + P^{(2)}(\mathbf{S}_{kl} + \mathbf{S}_{kl+1}) + H_{\Lambda_{k+1}}^f/2 \\ &= H_{\Lambda_k \cup \Lambda_{k+1}}^f/2 + P^{(2)}(\mathbf{S}_{kl} + \mathbf{S}_{kl+1})/2\end{aligned}$$

(with the convention $n+1 \equiv 1$). Clearly, $\text{Ker}(\overline{H}_{k,k+1}) = \text{Ker}(H_{\Lambda_k \cup \Lambda_{k+1}}^f) = \mathcal{G}_{\Lambda_k \cup \Lambda_{k+1}}$ and

$$\overline{H}_{k,k+1} \geq H_{\Lambda_k \cup \Lambda_{k+1}}^f/2 \geq \gamma/2(1 - G_{\Lambda_k \cup \Lambda_{k+1}}). \quad (27)$$

Now, an important role is played by the asymptotic commutativity of the projectors $G_{\Lambda_k \cup \Lambda_{k+1}}$, which we utilize as follows. For each k we orthogonalize the basis $\Omega_{\Lambda_k;ab}$:

$$\Omega'_{\Lambda_k;ab} = \sum_{c,d=1}^2 (g_{\Lambda_k}^{-1/2})_{ab,cd} \Omega_{\Lambda_k;cd}$$

and next define

$$\Omega''_{\Lambda_k \cup \Lambda_{k+1};ab} = \Omega'_{\Lambda_k;a1} \otimes \Omega'_{\Lambda_{k+1};2b} - \Omega'_{\Lambda_k;a2} \otimes \Omega'_{\Lambda_{k+1};1b}.$$

Denote by $G''_{k,k+1}$ the projector onto the four-dimensional subspace spanned by $\Omega''_{\Lambda_k \cup \Lambda_{k+1};ab}$ in $\mathcal{H}_{\Lambda_k \cup \Lambda_{k+1}}$. A straightforward calculation shows then that $G''_{k-1,k}$ commutes with $G''_{k,k+1}$. At the same time, by the property 2) above

$$\|G''_{k,k+1} - G_{\Lambda_k \cup \Lambda_{k+1}}\| = O(\epsilon^l) \quad (28)$$

with some $\epsilon < 1$. Now we use the following abstract observation.

Lemma 5. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be three finite-dimensional Hilbert spaces, and H_1 and H_2 be two commuting self-adjoint operators acting on $\mathcal{H}_1 \otimes \mathcal{H}_3$ and $\mathcal{H}_3 \otimes \mathcal{H}_2$, respectively. Then there exists a decomposition*

$$\mathcal{H}_3 = \oplus_s (\mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2)$$

such that for each s $\mathcal{H}_1 \otimes \mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2$ and $\mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2 \otimes \mathcal{H}_2$ are invariant subspaces of H_1 and H_2 , respectively, and, furthermore, the restriction $H_1|_{\mathcal{H}_1 \otimes \mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2}$ is an operator acting only on $\mathcal{H}_1 \otimes \mathcal{H}_{3,s}^1$, and the restriction $H_2|_{\mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2 \otimes \mathcal{H}_2}$ is an operator acting only on $\mathcal{H}_{3,s}^2 \otimes \mathcal{H}_2$.

Proof. Decompose $H_1 = \sum_i H_{11,i} \otimes H_{13,i}$, where $H_{11,i}$ are linearly independent operators on \mathcal{H}_1 , and $H_{13,i}$ are operators on \mathcal{H}_3 . Consider the algebra $\mathcal{A}_1 \subset \mathcal{B}(\mathcal{H}_3)$ generated by the operators $H_{13,i}$ and the unity. \mathcal{A}_1 is a von Neumann algebra (closed under taking adjoints). Similarly, we decompose $H_2 = \sum_i H_{23,i} \otimes H_{22,i}$, where $H_{22,i} \in \mathcal{B}(\mathcal{H}_2)$ are linearly independent and

$H_{23,i} \in B(\mathcal{H}_3)$. By the commutativity assumption the operators $H_{23,i}$ lie in the commutant \mathcal{A}'_1 of \mathcal{A}_1 . But it follows from the well-known classification of finite-dimensional von Neumann algebras (see e.g. [7]) that there exists a decomposition

$$\mathcal{H}_3 = \oplus_s (\mathcal{H}_{3,s}^1 \otimes \mathcal{H}_{3,s}^2) \quad (29)$$

such that

$$\mathcal{A}_1 = \oplus_s (\mathcal{B}(\mathcal{H}_{3,s}^1) \otimes \mathbf{1}_{\mathcal{H}_{3,s}^2}), \quad \mathcal{A}'_1 = \oplus_s (\mathbf{1}_{\mathcal{H}_{3,s}^1} \otimes \mathcal{B}(\mathcal{H}_{3,s}^2)).$$

Therefore (29) is the desired decomposition. \square

We apply this observation to the Hilbert spaces $\mathcal{H}_{\Lambda_{k-1}}, \mathcal{H}_{\Lambda_k}, \mathcal{H}_{\Lambda_{k+1}}$ and operators $G''_{k-1,k}, G''_{k,k+1}$. The relevant decomposition then is

$$\mathcal{H}_{\Lambda_k} = \mathcal{F}_k^1 \otimes \mathcal{F}_k^2 \oplus (\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k}). \quad (30)$$

Here $\mathcal{F}_k^1, \mathcal{F}_k^2$ are two-dimensional Hilbert spaces such that $\mathcal{F}_k^1 \otimes \mathcal{F}_k^2 = \mathcal{G}_{\Lambda_k}$. One can choose orthonormal bases $\{v_{k,a}^1\}_{a=1,2}$ and $\{v_{k,b}^2\}_{b=1,2}$ in \mathcal{F}_k^1 and \mathcal{F}_k^2 so that $\Omega'_{\Lambda_k;ab} = v_{k,a}^1 \otimes v_{k,b}^2$. The subspace $\mathcal{H}_{\Lambda_{k-1}} \otimes \mathcal{G}_{\Lambda_k}$ is invariant for $G''_{k-1,k}$, and in this subspace $G''_{k-1,k}$ essentially acts only on $\mathcal{H}_{\Lambda_{k-1}} \otimes \mathcal{F}_k^1$; similarly, $\mathcal{G}_{\Lambda_k} \otimes \mathcal{H}_{\Lambda_{k+1}}$ is invariant for $G''_{k,k+1}$, and it acts there only on $\mathcal{F}_k^2 \otimes \mathcal{H}_{\Lambda_{k+1}}$. The subspaces $\mathcal{H}_{\Lambda_{k-1}} \otimes (\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k})$ and $(\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k}) \otimes \mathcal{H}_{\Lambda_{k+1}}$ lie in the kernels of $G''_{k-1,k}, G''_{k,k+1}$, respectively (hence we don't factor $\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k}$ in (30)).

If we use the decomposition (30) for two neighboring blocks k and $k+1$, we get a decomposition of $\mathcal{H}_{\Lambda_k} \otimes \mathcal{H}_{\Lambda_{k+1}}$ as a direct sum of four subspaces:

$$\begin{aligned} \mathcal{H}_{\Lambda_k} \otimes \mathcal{H}_{\Lambda_{k+1}} &= (\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k}) \otimes (\mathcal{H}_{\Lambda_{k+1}} \ominus \mathcal{G}_{\Lambda_{k+1}}) \\ &\oplus (\mathcal{H}_{\Lambda_k} \ominus \mathcal{G}_{\Lambda_k}) \otimes \mathcal{F}_{k+1}^1 \otimes \mathcal{F}_{k+1}^2 \\ &\oplus \mathcal{F}_k^1 \otimes \mathcal{F}_k^2 \otimes (\mathcal{H}_{\Lambda_{k+1}} \ominus \mathcal{G}_{\Lambda_{k+1}}) \\ &\oplus \mathcal{F}_k^1 \otimes \mathcal{F}_k^2 \otimes \mathcal{F}_{k+1}^1 \otimes \mathcal{F}_{k+1}^2. \end{aligned}$$

The first three subspaces lie in the kernel of $G''_{k,k+1}$, whereas in the fourth $G''_{k,k+1}$ acts as the projector onto the vector

$$v_{k,k+1} \equiv v_{k,1}^2 \otimes v_{k+1,2}^1 - v_{k,2}^2 \otimes v_{k+1,1}^1 \quad (31)$$

in the space $\mathcal{F}_k^2 \otimes \mathcal{F}_{k+1}^1$.

In order to get a classical model in the sense of Introduction we introduce additional Hilbert spaces $\mathcal{F}_k^3, \mathcal{F}_k^4$, with $\dim \mathcal{F}_k^3 = \dim \mathcal{F}_k^4 = 3^{l/2} - 2$ (assuming that l is even), so that

$$\mathcal{H}_{\Lambda_k} = (\mathcal{F}_k^1 \oplus \mathcal{F}_k^3) \otimes (\mathcal{F}_k^2 \oplus \mathcal{F}_k^4).$$

Now define new Hilbert spaces $\overline{\mathcal{H}}_{k,k+1}$ by

$$\overline{\mathcal{H}}_{k,k+1} = (\mathcal{F}_k^2 \oplus \mathcal{F}_k^4) \otimes (\mathcal{F}_{k+1}^1 \oplus \mathcal{F}_{k+1}^3).$$

The initial scaled spin chain with sites indexed by k and Hilbert spaces \mathcal{H}_{Λ_k} assigned to k is then equivalent to the chain with sites indexed by pairs $(k, k+1)$ and Hilbert spaces $\overline{\mathcal{H}}_{k,k+1}$ assigned to the new sites $(k, k+1)$. Let $h_{k,k+1}$ be the projector in $\overline{\mathcal{H}}_{k,k+1}$ onto the orthogonal complement to the vector $v_{k,k+1}$ introduced in (31). Consider the operator

$$H_{\Lambda,0} = 3l \sum_{k=1}^n h_{k,k+1}.$$

We claim that if l is large enough (independently of n), then this operator is the desired classical Hamiltonian, such that the AKLT Hamiltonian H_{Λ}^p is its relatively bounded perturbation satisfying assumptions of Theorem 2.

To prove this, we write H_{Λ}^p as

$$H_{\Lambda}^p = H_{\Lambda,0} + \sum_{k=1}^n \phi_{k,k+1}^{(r)} + \sum_{k=1}^n \phi_{k,k+1}^{(b)},$$

where

$$\phi_{k,k+1}^{(r)} = (\mathbf{1} - G''_{k,k+1}) \overline{H}_{k,k+1} (\mathbf{1} - G''_{k,k+1}) - l(h_{k-1,k} + h_{k,k+1} + h_{k+1,k+2})$$

will be the “purely relatively bounded” part of the perturbation, and

$$\phi_{k,k+1}^{(b)} = \overline{H}_{k,k+1} - (\mathbf{1} - G''_{k,k+1}) \overline{H}_{k,k+1} (\mathbf{1} - G''_{k,k+1})$$

the bounded part. First we estimate $\|\phi_{k,k+1}^{(b)}\|$:

$$\begin{aligned} \|\phi_{k,k+1}^{(b)}\| &= \|(\mathbf{1} - G_{\Lambda_k \cup \Lambda_{k+1}}) \overline{H}_{k,k+1} (\mathbf{1} - G_{\Lambda_k \cup \Lambda_{k+1}}) \\ &\quad - (\mathbf{1} - G''_{k,k+1}) \overline{H}_{k,k+1} (\mathbf{1} - G''_{k,k+1})\| \\ &\leq \|G_{\Lambda_k \cup \Lambda_{k+1}} - G''_{k,k+1}\| (\|\overline{H}_{k,k+1} (\mathbf{1} - G_{\Lambda_k \cup \Lambda_{k+1}})\| \\ &\quad + \|(\mathbf{1} - G''_{k,k+1}) \overline{H}_{k,k+1}\|) \\ &= O(l\epsilon^l), \end{aligned} \tag{32}$$

by (28) and because $\|\overline{H}_{k,k+1}\| \leq l$. Now we analyze the term $\sum_k \phi_{k,k+1}^{(r)}$. We claim that the condition (6) holds with $\alpha = -(1 - \gamma/(6l) + O(\epsilon^l))$, uniformly for all $I \subset \{1, 2, \dots, n\}$. Indeed, note first that

$$h_{k,k+1} \leq \mathbf{1} - G''_{k,k+1} \leq h_{k-1,k} + h_{k,k+1} + h_{k+1,k+2}. \tag{33}$$

It follows from the right inequality that $\phi_{k,k+1}^{(r)} \leq 0$. Therefore the maximum over I in (6) is attained when $I = \Lambda$. By (27),(28) and the left inequality in (33),

$$\begin{aligned} & (1 - G''_{k,k+1}) \overline{H}_{k,k+1} (1 - G''_{k,k+1}) \\ & \geq \gamma/2 (1 - G''_{k,k+1}) (1 - G_{\Lambda_k \cup \Lambda_{k+1}}) (1 - G''_{k,k+1}) \\ & \geq (\gamma/2 + O(\epsilon^l)) (1 - G''_{k,k+1}) \\ & \geq (\gamma/2 + O(\epsilon^l)) h_{k,k+1} \end{aligned}$$

and hence

$$\begin{aligned} \sum_{k=1}^n \phi_{k,k+1}^{(r)} & \geq \sum_{k=1}^n ((\gamma/2 + O(\epsilon^l)) h_{k,k+1} - l(h_{k-1,k} + h_{k,k+1} + h_{k+1,k+2})) \\ & = -(1 - \gamma/(6l) + O(\epsilon^l)) H_{\Lambda,0}. \end{aligned}$$

Since $\phi_{k,k+1}^{(r)} \leq 0$, this proves our claim about relative boundedness with $\alpha = 1 - \gamma/(6l) + O(\epsilon^l)$.

Now we apply Theorem 2. We have $(1 - \alpha)^{\varkappa(1+\nu)} = (\gamma/(6l) + O(\epsilon^l))^{2\varkappa}$. On the other hand, by (32), the bounded part of the perturbation is $O(l\epsilon^l)$, which is asymptotically less than $(\gamma/(6l) + O(\epsilon^l))^{2\varkappa}$. Therefore for l large enough H_{Λ}^p is a relatively bounded perturbation of $H_{\Lambda,0}$ so that Theorem 2 is applicable. The conclusion of Theorem 3 follows now from Theorem 2, because a sufficiently weak perturbation of the AKLT model remains within the range of perturbations of $H_{\Lambda,0}$, where Theorem 2 is applicable.

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